Assignment 11

1. Approximate a solution to the wave equation with four steps in time if the boundary conditions are $u_a(t) = \sin(t)$ and $u_b(t) = -\sin(t)$ and the initial states are $u_0(x) = \sin(\pi x)$ and $u_0^{(1)}(x) = 0$ if the interval in space is [0, 1] and $h = 0.2$. The wave speed is $c = 2$. You should use the Δt found in the course notes to ensure convergence.

2. What is c for electromagnetic waves? You may have to include an assumption.

The speed of light in a vacuum.

3. Approximate a solution to Laplace's equation if we have a square region with sides $[0, 1] \times [0, 1]$ with $h = 0.25$ and two opposite walls are charged to 0 V and the other walls are at 5 V. What changes if the boundary point above the bottom right corner is changed to 100 V?

```
% The boundary values are
%
%      2.5   0    0    0   2.5
%
%       5    x    x    x    5
%
%       5    x    x    x    5
%
%       5    x    x    x    5
%
%      2.5   0    0    0   2.5
%
```

```
% We can label the points at which we have unknowns:
%
%      2.5   0    0    0   2.5
%
%       5    v1   v2   v3   5
%
%       5    v4   v5   v6   5
%
%       5    v7   v8   v9   5
%
%      2.5   0    0    0   2.5
%
% At each unknown, the value must be equal to the average of the points
% around it; hence, for example, we have at v1:
%                 0 + 5 + v2 + v4
%           v1 = -----------------
%                        4
% We can rewrite this in the more standard form:
%              4v1 - v2 - v4 = 5
%
% Doing this at all nine unknown points:
%                   4v1 - v2 - v4 = 5
%              4v2 - v1 - v3 - v5 = 0
%                   4v3 - v2 - v6 = 5
%              4v4 - v1 - v5 - v7 = 5
%         4v5 - v2 - v4 - v6 - v8 = 0
%              4v6 - v3 - v5 - v9 = 5
%                   4v7 - v4 - v8 = 5
%              4v8 - v5 - v7 - v9 = 0
%                   4v9 - v6 - v8 = 5
%
% This defines the system of linear equations described by:
%
A = [4 -1  0 -1  0  0  0  0  0
    -1  4 -1  0 -1  0  0  0  0
     0 -1  4  0  0 -1  0  0  0
    -1  0  0  4 -1  0 -1  0  0
     0 -1  0 -1  4 -1  0 -1  0
     0  0 -1  0 -1  4  0  0 -1
     0  0  0 -1  0  0  4 -1  0
     0  0  0  0 -1  0 -1  4 -1
     0  0  0  0  0 -1  0 -1  4];

b = [5 0 5 5 0 5 5 0 5]';
```

```
% Solving this, we get:
A \ b
   2.5
   1.8754999999999999
   2.5
   3.125
   2.5
   3.125
   2.5
   1.8750000000000001
   2.5
```

Note the two small, almost imperceptible, numeric errors.

```
%       2.5    0      0      0      2.5
%
%       5      2.5    1.875  2.5    5
%
%       5      3.125  2.5    3.125  5
%
%       5      2.5    1.875  2.5    5
%
%       2.5    0      0      0      2.5
```

You will note from the symmetry that many values are the same.

If the point above the bottom right corner had a voltage of 100, this would only change the vector:

```
%      2.5   0    0    0   2.5
%
%       5    v1   v2   v3   5
%
%       5    v4   v5   v6   5
%
%       5    v7   v8   v9  100  <-- a voltage potential of 100 V
%
%      2.5   0    0    0   2.5
```

This changes the last equation, so it only changes the last entry of the vector:

```
%                    4v9 - v6 - v8 = 100
b = [5 0 5 5 0 5 5 0 100]';
format
A \ b
    3.7723
    4.4196
    5.4687
    5.6696
    8.4375
   12.4554
    5.4688
   11.2054
   30.9152
```

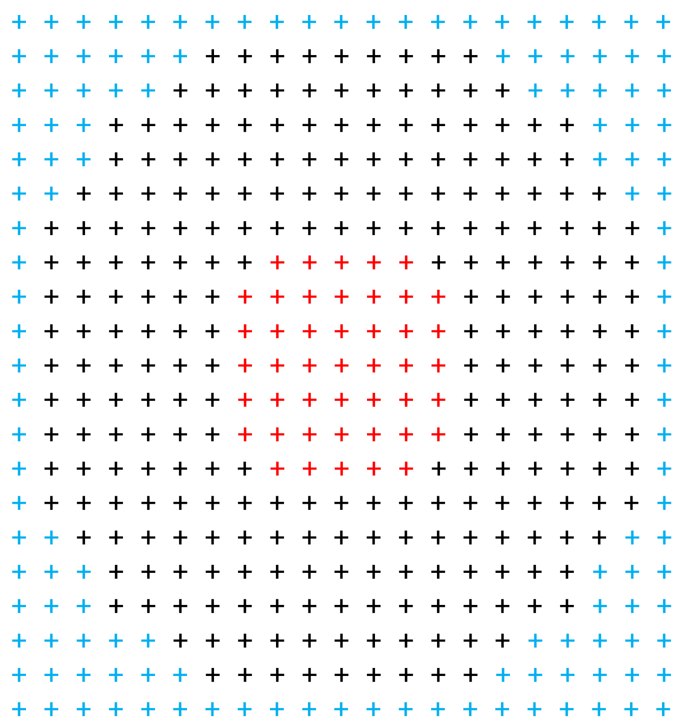
Now the approximations of the corresponding temperatures are:

```
%       2.5      0        0        0        2.5
%
%       5        3.7723   4.4196   5.4687   5
%
%       5        5.6696   8.4375  12.4554   5
%
%       5        5.4688  11.2054  30.9152 100
%
%       2.5      0        0        0        2.5
```

4. Approximate a solution to Laplace's equation if we have a square region with sides $[0, 1] \times [0, 1]$ with $h = 0.25$ and two opposite walls are charged to 0 V, one intermediate wall is at 5 V and the last wall is insulated.

5. How would you propose setting up the boundary conditions if you have a cross section of a coaxial cable that has a radius of 1.0 with an inner cable that has a radius of 0.4. The outside of the cable is kept at 0 V and the inner conducting cable is kept at 5 V. You should use an $h = 0.1$, but you don't have to explicitly set up the system of equations; instead, just indicate which points in the grid are associated with the 0 V boundary condition, which are associated with the 5 V boundary condition, and which are unknown and must be solved for.

As an example, you could have a 21 × 21 grid, where the point $x_j = 0.1j$ and $y_k = 0.1k$, and where the distance from the origin is given by $\sqrt{(0.1j)^2 + (0.1k)^2} = 0.1\sqrt{j^2 + k^2}$. Thus, any point with a distance equal to or greater than one would be assigned 0 V, and any point less than or equal to 0.4 would be assigned 5 V. This leaves all the black "plus" symbols as unknowns to be solved for. There are 256 such points, so these could be assigned unknowns from u_1 to u_{256} and then each point would describe one equation.



6. Approximate a solution to Laplace's equation if we have a three-dimensional orthotope region with sides $[0, 1.25] \times [0, 1] \times [0, 1]$ with $h = 0.25$ and one square wall is at 0 V while the other five walls are insulated with the exception of a single point in the center of the opposite square wall that is kept at 5 V.

7. If you consider the solution to Question 6, you will realize that some points, by symmetry, must have the same value. Right now, we have a system of 36 equations in 36 unknowns. Use symmetry (identifying points that must have the same value) to reduce this to 12 equations in 12 unknowns.

8. Do the dimensions of the object change the result of Laplace's equation? For example, if the square region in Questions 3 and 4 were 1 cm or 1 km on each side, would this affect the approximations at the points, assuming h was also scaled appropriate?

No, the solution to Laplace's equation is *scale invariant*. The approximation simply says that at each point, that value is the average of the values around it. It does not matter if $h = 0.1$ mm, $h = 1$ m, or $h = 15.42$ km.

Contrast this with the heat-equation: If you had to hold at one end a 1 cm diameter rod that is either 20 cm long or 100 cm long, and the other end would have a blow-torch applied to it, which do you think you could hold longer?

9. Find an approximation for the minimum of the polynomial $x^4 + x^2 - 40x + 400$ using the step-by-step optimization routine described starting with $x_0 = 0$ and an initial $h = 1$, continuing until $h < 0.5$.

Let this polynomial be represented by p. We note that $p(-1) = 442$, $p(0) = 400$ and $p(1) = 362$, so we move forward: $p(2) = 340$ and $p(3) = 370$, so we choose $x_1 = 2$ and reduce h by half.

Thus, we $p(1.5) = 347.3125$, $p(2) = 340$ and $p(2.5) = 345.3125$, so we choose $x_2 = 2$ and reduce h by half.

At this point, $h = 0.25$, and thus, our approximation of the minimum would be 340, and the actual minimum value is approximately 339.8439121.

If we were to apply a few more steps, we would continue to reduce h by half to find that

$$p(1.75) = 342.44140625,\ p(2) = 340 \text{ and } p(2.25) = 340.69140625,$$

so $x_3 = 2$; and then

$$p(1.875) = 340.875244140625,\ f(2) = 340,\ f(2.125) = 339.906494140625$$

so $x_4 = 2.125$. The minimum value 339.9064940625 has a percent relative error of only 0.25%.

10. Find an approximation for the minimum of the polynomial $x^4 + x^2 - 40x + 400$ by applying two steps of Newton's method for finding extrema starting with $x_0 = 2$.

```
>>   f = @(x)(   x^4 +   x^2 - 40*x + 400);
>>  df = @(x)( 4*x^3 + 2*x   - 40);
>> ddf = @(x)(12*x^2 + 2);
>> x0 = 2
    x0 =  2
>> f(x0)
        340
>> x1 = x0 - df(x0)/ddf(x0)
    x1 =  2.08
>> f(x1)
        339.84413696
>> x2 = x1 - df(x1)/ddf(x1)
    x2 =  2.077113181791205
>> f(x2)
        339.8439120961672
>> x3 = x2 - df(x2)/ddf(x2)
    x3 =  2.077109315274261
>> f(x3)
        339.8439120957652
```

11. Find an approximation for the minimum of the function $-\sin(x) + \sin(2x)$ using the step-by-step optimization routine described starting with $x_0 = 0$ and an initial $h = 1$, continuing until $h < 0.5$.

12. Find an approximation for the minimum of the function $-\sin(x) + \sin(2x)$ by applying two steps of Newton's method for finding extrema starting with $x_0 = 2$.

```
  f = @(x)(-sin(x) + sin(2*x));
 df = @(x)(-cos(x) + 2*cos(2*x));
ddf = @(x)( sin(x) - 4*sin(2*x));
x0 = 2
     x0 =  2
f(x0)
          -1.666099922133610
x1 = x0 - df(x0)/ddf(x0)
     x1 =  2.226378439770254
f(x1)
          -1.759177686707815
x2 = x1 - df(x1)/ddf(x1)
     x2 =  2.205734868878529
f(x2)
          -1.760172581166689
x3 = x2 - df(x2)/ddf(x2)
     x3 =  2.205663198915996
f(x3)
          -1.760172593046087
```

Acknowledgement: Jackson Toth for noting the solution used 48*x* and not 40*x*. Tony Yu for observing that the arguments of the polynomials were wrong, even if the values were correct.